Some remarks on a model for rate-independent damage in thermo-visco-elastodynamics

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Abstract. This note deals with the analysis of a model for partial damage, where the rate-independent, unidirectional flow rule for the damage variable is coupled with the rate-dependent heat equation, and with the momentum balance featuring inertia and viscosity according to Kelvin-Voigt rheology. The results presented here combine the approach from Roubíček [1, 2] with the methods from Lazzaroni/Rossi/Thomas/Toader [3]. The present analysis encompasses, differently from [2], the monotonicity in time of damage and the dependence of the viscous tensor on damage and temperature, and, unlike [3], a nonconstant heat capacity and a time-dependent Dirichlet loading.

1. Introduction – Energetic solutions for rate-independent processes coupled with rate-dependent effects

In this note we discuss the existence of solutions for an evolutionary model of partial damage, where the rate-independent, unidirectional flow rule for the damage variable is coupled with the rate-dependent heat equation, and with the momentum balance featuring inertia and viscosity according to Kelvin-Voigt rheology. Systems with a mixed rate-independent/rate-dependent character were considered in [1] in the isothermal case, where a suitable notion of weak (*energetic*) solution was introduced, and then in [2] where this notion was extended to thermal processes. In the latter paper, a general existence result for *energetic solutions* was proved, with application to a wide class of thermo-viscoelastic material systems in the frame of *generalized standard solids*, where the flow rule for the internal variable has rate-independent character. The damage model treated here pertains to this class: the internal variable z assesses the soundness of the material, so that one will have $z = 1$ in the fully undamaged, and $z = 0$ in the completely damaged cases, respectively. Additionally, in the model discussed here, z also affects the elastic and the viscous stress tensors.

Here we will further contribute to the analysis initiated in [2] by pointing out that the existence result therein can be extended to the case in which the evolution for the internal variable is *unidirectional* (i.e., monotone nonincreasing), as in the context of the damage model presently considered. Moreover, we will show that time-dependent Dirichlet loadings for the displacement variable can be encompassed in the analysis, whereas in [2, 3] the momentum equation was supplemented with zero Dirichlet boundary data. In this note we will particularly focus on the techniques to treat the difficulties resulting from these additional features of the model. We refer to our previous work [3] for a detailed survey of the literature on rate-independent and rate-dependent damage models in (thermo-)viscoelasticity and for a more detailed discussion of the PDE system under consideration.

We will prove the existence of *energetic solutions* for the damage system using a time discretization, by now standard within the analysis of rate-independent problems. We will also show that, under appropriate conditions on the nonlinear constitutive functions featuring in the PDE system, the time discrete scheme can be *fully decoupled*, which might turn out to be interesting towards the numerical investigation of this model.

The PDE system. We consider the following PDE system which describes the behavior of a thermo-visco-elastic body subject to damage; it consists of the momentum balance (1a), the flow rule (1b), and the heat equation (1c):

$$\rho\ddot{u} - \mathrm{div}\,(\mathbb{D}(z,\theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta\,\mathbb{B}) = f_{\mathrm{V}} \qquad \text{in } (0,T)\times\Omega\,, \tag{1a}$$

$$\partial\mathrm{R}_1(\dot{z}) + \mathrm{D}_z G(z,\nabla z) - \mathrm{div}\,(\mathrm{D}_\xi G(z,\nabla z)) + \tfrac{1}{2}\mathbb{C}'(z)e(u):e(u) \ni 0 \qquad \text{in } (0,T)\times\Omega\,, \tag{1b}$$

$$c_{\mathrm{v}}(\theta)\dot{\theta} - \mathrm{div}\,(\mathbb{K}(z,\theta)\nabla\theta) = \mathrm{R}_1(\dot{z}) + (\mathbb{D}(z,\theta)e(\dot{u}) - \theta\,\mathbb{B}):e(\dot{u}) + H \qquad \text{in } (0,T)\times\Omega\,. \tag{1c}$$

Here the unknowns (u, z, θ) stand for the displacement vector field, the damage variable, and the absolute temperature, respectively, $(0, T)$ indicates the time interval, while Ω is a bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ representing the reference configuration. The strain tensor is $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, the constant $\rho > 0$ is the mass density, $\mathbb{D}(z, \theta)$ and $\mathbb{C}(z)$ are the viscous and the elastic tensors, respectively. Thermal stresses are featured by $\theta\,\mathbb{B}$ with $\mathbb{B}$ a fixed symmetric matrix coupling the momentum and the heat equations. In (1c), $c_{\mathrm{v}}(\theta)$ and $\mathbb{K}(z, \theta)$ are, respectively, the heat capacity and the heat conductivity of the material. In (1b) the term $G(z, \nabla z)$ is a regularization for the damage variable as it involves its gradient (e.g. in Sobolev-sense). The term $\mathrm{R}_1(\dot{z})$ is a 1-homogeneous dissipation potential which, in order to encode the rate-independence and the unidirectionality of the damage process, is chosen as

$$\mathrm{R}_1(v) := \begin{cases} |v| & \text{if } v \le 0\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{2}$$

cf. also e.g. [4] for this ansatz. Note that the unidirectionality is reflected by the fact that $\mathrm{R}_1(\dot{z}) = \infty$ if $\dot{z} > 0$. This ensures that a solution z will be nonincreasing in time in accordance with the definition of z; recall that $z(x) = 1$ in sound and $z(x) = 0$ in completely damaged material points x. In the flow rule (1b) the symbol ∂ indicates the subdifferential in the sense of convex analysis while D_z and D_ξ denote the Gâteaux derivatives. The external forces and the heat source are denoted by f_{V} and H, respectively.

The PDE system (1) is supplemented with Cauchy conditions given on $u(0)$, $\dot{u}(0)$, $z(0)$, and $\theta(0)$, and with the boundary conditions

$$(\mathbb{D}(z,\theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta\,\mathbb{B})\,\nu = f_{\mathrm{S}} \qquad \text{on } (0,T)\times\partial_{\mathrm{N}}\Omega\,, \tag{3a}$$

$$u = g \qquad \text{on } (0,T)\times\partial_{\mathrm{D}}\Omega\,, \tag{3b}$$

$$\mathrm{D}_\xi G(z,\nabla z)\,\nu = 0 \qquad \text{on } (0,T)\times\partial\Omega\,, \tag{3c}$$

$$\mathbb{K}(z,\theta)\nabla\theta\cdot\nu = h \qquad \text{on } (0,T)\times\partial\Omega\,, \tag{3d}$$

where $\partial_{\mathrm{D}}\Omega$ and $\partial_{\mathrm{N}}\Omega := \partial\Omega\backslash\partial_{\mathrm{D}}\Omega$ are the Dirichlet and the Neumann part of the boundary, ν denotes the outer unit normal vector to $\partial\Omega$, and f_{S}, g, and h are prescribed external data depending on time and space. In Section 2 we shall detail the assumptions on the constitutive functions and on the given data featuring in system (1).

To some extent, system (1) is a particular case of the general model for rate-independent processes in thermo-viscoelastic materials proposed in [2]. In particular, as in the latter paper the heat equation features a nonconstant heat capacity, depending on θ in a nonlinear way, and accordingly it requires appropriate treatment. Indeed, it will turn out to be useful to switch from the temperature to the *enthalpy* variable, defined as a primitive of the (positive) function c_{v}. Moreover, as in [2], we shall have to impose specific growth conditions on c_{v} and on the matrix of heat conduction coefficients $\mathbb{K}$ in order to handle the quadratic terms on the right-hand side of (1c). We refer to [5] for a thermodynamical derivation of the growth assumptions used in what follows.

Nonetheless, let us highlight that, differently from [2] our analysis also encompasses the unidirectionality in the evolution for z, the dependence of the viscous tensor $\mathbb{D}$ on z and θ, and, differently from [3], also a time-dependent Dirichlet loading g.

Indeed, a model for rate-independent, unidirectional, partial damage in thermo-viscoelastic materials with inertia was also proposed in [3] in the case of *constant* heat capacity c_{v}, which allowed us to avoid the enthalpy transformation. The assumption of a constant heat capacity is valid for large values of the temperature, while a nonconstant heat capacity describes low-temperature regimes. We remark that in [3] it was possible to only *partially* decouple the time-discrete scheme, i.e., only the approximate flow rule for the internal variable could be decoupled from the other equations. Moreover, time-dependent boundary conditions on the displacement u could be accounted for only if the viscous tensor was assumed to be constant.

In the following lines, we briefly sketch the main points of the analysis. First of all, let us remark that, as in [3], we shall have to resort to a weak notion of solution for the initial-boundary value problem for system (1), introduced in [1, 2] and hereafter referred to as *energetic*. While postponing all details to Definition 2.1, we may mention here that this concept consists of the unidirectionality and semistability conditions for the damage variable z combined with a (mechanical) energy balance, and coupled with the weak (distributional-type) formulations of the momentum and enthalpy equations.

After stating our working assumptions on the problem data and introducing energetic solutions, in Section 2 we will state our main existence result, Theorem 2.2. Its proof will be developed throughout Section 3, according to the general strategy suggested in [2]. In fact, in this contribution we will only sketch some parts of the proof, referring for certain details to the latter paper, as well as to [3] for the handling of *unidirectional* processes. Instead, we will dwell on the techniques allowing us to fully decouple the time-discrete scheme and to account for the dynamics of the Dirichlet loading, which causes additional rate-dependent energy terms, see the comments in Section 3.

2. Setup and main result

In this section we collect the conditions on the constitutive functions featuring in system (1), as well as on the loadings and on the prescribed external and initial data. These functions also enter in the free energy ψ of the system which has the structure

$$\psi(e(u), z, \nabla z) = \varphi(e(u), z, \nabla z) + \theta\,\phi(e(u)) - \phi_0(\theta)\,,$$

considered in [2]. The purely thermal contribution ϕ_0 determines the heat capacity function c_{v} via $c_{\mathrm{v}}(\theta) := \theta\,{\phi_0}''(\theta)$. As for the mechanical part φ, in the present case we take it in the form

$$\varphi(e(u), z, \nabla z) := \tfrac{1}{2}\mathbb{C}(z)e(u) : e(u) + G(z, \nabla z)\,,$$

while $\phi(e(u)) := -\mathbb{B} : e(u)$.

Next, we derive the version of (1) in terms of the *enthalpy* in place of the temperature, and for the resulting system we recall the notion of *energetic solution* from [2]. The statement of our existence result, Theorem 2.2, closes this section.

Assumptions on the heat capacity and heat conductivity. As mentioned in the introduction, the treatment of the heat equation relies on specific growth conditions on c_v and $\mathbb{K}$, adopted from [2] and further tailored to our existence analysis. More precisely, we assume that

$$\begin{aligned} &c_\mathrm{v} \in \mathrm{C}^0(\mathbb{R};\mathbb{R}^+) \text{ is such that} \\ &\quad \exists\, \alpha_1 \geq \alpha = 1\,,\ c_1 \geq c_0 > 0\ \forall\, \theta \in [0,+\infty): \quad c_0(1+\theta)^\alpha \leq c_\mathrm{v}(\theta) \leq c_1(1+\theta)^{\alpha_1}\,, \end{aligned} \tag{4a}$$

$$\begin{aligned} &\mathbb{K} \in \mathrm{C}^0(\mathbb{R}\times\mathbb{R};\mathbb{R}^{d\times d}) \text{ is symmetric and} \\ &\quad \exists\, c_2,\, c_3 > 0\ \forall\, (z,\theta) \in [0,1]\times[0,+\infty): \quad c_2\, c_\mathrm{v}(\theta) \leq |\mathbb{K}(z,\theta)| \leq c_3\, c_\mathrm{v}(\theta)\,. \end{aligned} \tag{4b}$$

Let us shortly compare the growth condition in (4a) with the one in [2, formulae (3.12b) and (4.22)]. There, it is required that $\alpha > (d-2)/(d+2)$ with d the space dimension, which would result in $\alpha > 1/5$ in the present three-dimensional context. Hence, $\alpha = 1$ in (4a) is a special case.

This particular choice is made in order to generate a square-root growth of the temperature in dependence of the enthalpy variable, cf. (16), which will be a crucial ingredient to handle the thermal expansion term in combination with time-dependent Dirichlet data. It will also play a key role in the analysis of the time-discrete version of (the enthalpy reformulation of) system (1). In particular, it will be at the core of the proof of Lemma 3.3, by means of which it is possible to have a *fully decoupled* scheme. Let us point out that the linear growth from below imposed on the heat capacity in (4a) is also in accordance with [5, 6] in the context of small-strain thermo-viscoelasticity, where the heat capacity is assumed to be an affine function of temperature. In particular, see [5] for a thermodynamical derivation.

Assumptions on the material tensors. We require that the tensors $\mathbb{B} \in \mathbb{R}^{3\times3}$, $\mathbb{C}\colon \mathbb{R} \to \mathbb{R}^{3\times3\times3\times3}$, and $\mathbb{D}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{3\times3\times3\times3}$ fulfill

$$\mathbb{B} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}\,, \text{ and set } C_\mathbb{B} := |\mathbb{B}|\,, \tag{5a}$$

$$\mathbb{C} \in \mathrm{C}^{0,1}(\mathbb{R};\mathbb{R}^{3\times3\times3\times3}) \text{ and } \mathbb{D} \in \mathrm{C}^0(\mathbb{R}\times\mathbb{R};\mathbb{R}^{3\times3\times3\times3})\,, \tag{5b}$$

$$\mathbb{C}(z),\, \mathbb{D}(z,\theta) \in \mathbb{R}^{3\times3\times3\times3}_{\mathrm{sym}} \text{ are positive definite for all } z \in \mathbb{R}\,,\ \theta \in \mathbb{R}\,, \tag{5c}$$

$$\exists\, C^1_\mathbb{C},\, C^2_\mathbb{C} > 0\ \ \forall\, z \in \mathbb{R}\ \ \forall\, A \in \mathbb{R}^{3\times3}_{\mathrm{sym}}: \quad C^1_\mathbb{C}\,|A|^2 \leq \mathbb{C}(z)A : A \leq C^2_\mathbb{C}\,|A|^2\,, \tag{5d}$$

$$\exists\, C^1_\mathbb{D},\, C^2_\mathbb{D} > 0\ \ \forall\, z \in \mathbb{R}\ \ \forall\, \theta \in \mathbb{R}\ \ \forall\, A \in \mathbb{R}^{3\times3}_{\mathrm{sym}}: \quad C^1_\mathbb{D}\,|A|^2 \leq \mathbb{D}(z\,,\theta)A : A \leq C^2_\mathbb{D}\,|A|^2\,. \tag{5e}$$

In addition, we impose that $\mathbb{C}(\cdot)$ is monotonically nondecreasing, i.e.,

$$\forall\, A \in \mathbb{R}^{3\times3}_{\mathrm{sym}}\ \ \forall\, 0 \leq z_1 \leq z_2 \leq 1: \quad \mathbb{C}(z_1)A : A \leq \mathbb{C}(z_2)A : A\,. \tag{5f}$$

In the expressions above, $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ denotes the subset of symmetric matrices in $\mathbb{R}^{3\times3}$ and $\mathbb{R}^{3\times3\times3\times3}_{\mathrm{sym}}$ is the subset of symmetric tensors in $\mathbb{R}^{3\times3\times3\times3}$. In particular,

$$\mathbb{C}(z)_{ijkl}{=}\mathbb{C}(z)_{jikl}{=}\mathbb{C}(z)_{ijlk}{=}\mathbb{C}(z)_{klij} \ \text{ and } \ \mathbb{D}(z,\theta)_{ijkl}{=}\mathbb{D}(z,\theta)_{jikl}{=}\mathbb{D}(z,\theta)_{ijlk}{=}\mathbb{D}(z,\theta)_{klij}\,.$$

Assumptions on the damage regularization. We require that $G\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}\cup\{+\infty\}$ fulfills

Indicator: For every $(z,\xi)\in\mathbb{R}\times\mathbb{R}^3$: $\quad G(z,\xi)<+\infty \ \Rightarrow\ z\in[0,1]\,;$ (6a)

Continuity: $G\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}\cup\{+\infty\}$ is continuous on $\mathrm{dom}(G)$, and $G(0,0)=0\,;$ (6b)

Convexity: For every $z\in\mathbb{R}$, $G(z,\cdot)$ is convex; (6c)

Growth: There exist constants $q>1$ and $C_G^1, C_G^2>0$ such that for every $(z,\xi)\in\mathrm{dom}(G)$

$$C_G^1(|\xi|^q-1)\le G(z,\xi)\le C_G^2(|\xi|^q+1)\,. \tag{6d}$$

Accordingly, the state space $\mathcal{Z}$ is defined by

$$\mathcal{Z}:=\{z\in W^{1,q}(\Omega)\colon z\in[0,1] \text{ a.e. in } \Omega\}\,. \tag{7}$$

Assumptions on the given data. With a slight abuse of notation, we will denote by g also the extension into the domain of the non-zero Dirichlet boundary datum on the displacement. We require that

$$g\in H^1(0,T;H^1(\Omega;\mathbb{R}^3))\cap W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))\,. \tag{8}$$

On the initial data $(u_0,\dot u_0,z_0)$ we require

$$u_0\in\ H^1(\Omega;\mathbb{R}^3)\,,\quad u_0=g(0)\quad \text{on } \partial_{\mathrm{D}}\Omega\,,\quad \dot u_0\in\ L^2(\Omega;\mathbb{R}^3)\,,\quad z_0\in\mathcal{Z}\,, \tag{9}$$

and, in accordance with (4a), we impose on θ_0

$$\theta_0\in\ L^{\alpha_1+1}(\Omega)\,,\qquad \theta_0\ge 0\quad \text{a.e. in } \Omega\,, \tag{10}$$

where α_1 is the same as in (4a). On the loading and source terms f_{V}, f_{S}, H, and h we require

$$f_{\mathrm{V}}\in H^1(0,T;L^2(\Omega;\mathbb{R}^3))\,,\quad f_{\mathrm{S}}\in H^1(0,T;L^2(\partial_N\Omega;\mathbb{R}^3))\,, \tag{11a}$$

$$H\in L^2(0,T;L^2(\Omega))\,,\qquad H\ge 0 \text{ a.e. in } (0,T)\times\Omega\,, \tag{11b}$$

$$h\in L^2(0,T;L^2(\partial\Omega))\,,\qquad h\ge 0 \text{ a.e. in } (0,T)\times\partial\Omega\,. \tag{11c}$$

For later convenience, we also introduce $f:[0,T]\to H^1(\Omega;\mathbb{R}^3)^*$ defined by

$$\langle f(t),v\rangle_{H^1(\Omega;\mathbb{R}^3)}:=\int_\Omega f_{\mathrm{V}}(t)\cdot v\,\mathrm{d}x+\int_{\partial_N\Omega} f_{\mathrm{S}}\cdot v\,\mathrm{d}\mathcal{H}^2(x)\qquad \text{for all } v\in H^1(\Omega;\mathbb{R}^3)\,, \tag{12}$$

$\mathcal{H}^2$ denoting the 2-dimensional Hausdorff measure. It follows from (11a) that $f\in H^1(0,T;H^1(\Omega;\mathbb{R}^3)^*)$.

Observe that the requirements on g and on $f_{\mathrm{V}},\ldots,h$ could be slightly refined, cf. [2] and [3]. However, we choose to overlook this point to avoid overburdening the analysis with technicalities.

The enthalpy transformation. In view of the time-discretization procedure, it is useful to pass from the nonlinear term $c_{\mathrm{v}}(\theta)\dot\theta$ in (1c) to a linear one. This motivates the change of variables adopted in [2], by virtue of which we switch from the absolute temperature θ to the enthalpy variable w, defined via the so-called *enthalpy transformation*, viz.

$$w=\mathfrak{h}(\theta):=\int_0^\theta c_{\mathrm{v}}(s)\,\mathrm{d}s\,. \tag{13}$$

Thus, $\mathfrak{h}$ is a primitive function of $c_{\rm v}$, normalized in such a way that $\mathfrak{h}(0) = 0$. Since $c_{\rm v}$ is strictly positive (cf. assumption (4a) above), $\mathfrak{h}$ is strictly increasing. Thus, we define

$$\Theta(w) := \begin{cases} \mathfrak{h}^{-1}(w) & \text{if } w \geq 0\,, \\ 0 & \text{if } w < 0\,, \end{cases} \qquad \mathbb{J}(z,w) := \frac{\mathbb{K}(z,\Theta(w))}{c_{\rm v}(\Theta(w))}\,. \tag{14}$$

It follows from (4a) and (10) that

$$w_0 := \mathfrak{h}(\theta_0) \in L^1(\Omega) \quad \text{and} \quad w_0 \geq 0 \text{ a.e. in } \Omega\,, \tag{15}$$

and that

$$\exists\, C_\Theta^0,\, C_\Theta^1,\, C_\Theta^2 > 0 \;\forall\, w \in [0,+\infty): \quad C_\Theta^0 w^{1/(\alpha_1+1)} - C_\Theta^1 \leq \Theta(w) \leq C_\Theta^2 w^{1/2}\,, \tag{16}$$

whereas (4b) ensures that

$$c_2 \leq |\mathbb{J}(z,w)| \leq c_3 \quad \text{for all } (z,w) \in [0,1] \times [0,+\infty)\,. \tag{17}$$

In view of (13), and replacing all occurrences of θ by $\Theta(w)$, we obtain the enthalpy reformulation of system (1):

$$\rho\ddot{u} - \operatorname{div}\left(\mathbb{D}(z,\Theta(w))e(\dot{u}) + \mathbb{C}(z)e(u) - \Theta(w)\,\mathbb{B}\right) = f_{\rm V} \qquad \text{in } (0,T)\times\Omega\,, \tag{18a}$$

$$\partial\mathrm{R}_1(\dot{z}) + \partial_z G(z,\nabla z) - \operatorname{div}\left(\mathrm{D}_\xi G(z,\nabla z)\right) + \tfrac{1}{2}\mathbb{C}'(z)e(u) : e(u) \ni 0 \qquad \text{in } (0,T)\times\Omega\,, \tag{18b}$$

$$\dot{w} - \operatorname{div}\left(\mathbb{J}(z,w)\nabla w\right) = \mathrm{R}_1(\dot{z}) + \left(\mathbb{D}(z,\Theta(w))e(\dot{u}) - \Theta(w)\mathbb{B}\right) : e(\dot{u}) + H \quad \text{in } (0,T)\times\Omega\,. \tag{18c}$$

Energetic solutions. Let us now specify the notion of weak solution for system (18), supplemented with the boundary conditions (3). As already mentioned, along the lines of [2] the rate-independent flow rule for z is formulated by means of a semistability condition and of an energy balance, featuring the *mechanical* energy of the system

$$\mathcal{E}(t,u,z) := \int_\Omega \left(\tfrac{1}{2}\mathbb{C}(z)e(u) : e(u) + G(z,\nabla z)\right) \mathrm{d}x - \langle f(t), u\rangle_{H^1(\Omega;\mathbb{R}^3)}\,, \tag{19}$$

and the rate-independent dissipation potential

$$\mathcal{R}_1(\dot{z}) := \int_\Omega \mathrm{R}_1(\dot{z})\,\mathrm{d}x \tag{20}$$

with R_1 from (2). Observe that in the present case the mechanical energy equality (25) below shall reflect the nonhomogeneous boundary condition (3b). More specifically, the dynamics of the boundary loading g causes additional rate-dependent energy terms, cf. 3rd and 4th line of (25). Semistability and (mechanical) energy balance are coupled with the weak (distributional-type) formulations of the momentum and enthalpy equations. In particular, the enthalpy equation is weakly formulated with test functions in $W^{1,r'}(0,T;L^{r'}(\Omega)) \cap \mathrm{C}^0([0,T];W^{1,r'}(\Omega))$ for every $1 \leq r < \frac{5}{4}$. This requirement is tailored to the $L^r(0,T;W^{1,r}(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$-regularity for the enthalpy variable, which results from the Boccardo-Gallouët-type estimates developed in [2] (and only briefly hinted at in Section 3.3).

The right-hand side of the weakly formulated enthalpy equation will feature the total variation measure $|\dot{z}|$ of z (i.e., the heat produced by the rate-independent dissipation), which is defined on every closed set of the form $A := [t_1,t_2] \times C \subset [0,T] \times \overline{\Omega}$ by

$$|\dot{z}|\,(A) := \int_C \mathrm{R}_1(z(t_2) - z(t_1))\,\mathrm{d}x\,.$$

Definition 2.1. Given initial data $(u_0, \dot{u}_0, z_0)$ satisfying (9), and θ_0 complying with (10) (so that $w_0 = \mathfrak{h}(\theta_0)$ fulfills (15)), we call a triple (u, z, w) an *energetic solution* to system (18), supplemented with the boundary conditions (3), if the functions (u, z) have the regularity

$$\begin{aligned}&u \in H^1(0,T;H^1(\Omega;\mathbb{R}^3)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))\,,\\ &\text{with } u(t) = g(t) \text{ on } \partial_{\mathrm{D}}\Omega \quad \text{for all } t \in [0,T]\,,\end{aligned} \tag{21a}$$

$$\begin{aligned}&z \in L^\infty(0,T;W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega) \cap \mathrm{BV}([0,T];L^1(\Omega))\,,\ \text{cf. (6d)},\\ &z(t,x) \in [0,1] \text{ for a.a. } (t,x) \in (0,T)\times\Omega\,,\end{aligned} \tag{21b}$$

while

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*) \tag{21c}$$

for every $1 \le r < \frac{5}{4}$; (u, z, w) satisfy the initial conditions

$$u(0) = u_0\,, \quad \dot{u}(0) = \dot{u}_0\,, \quad z(0) = z_0\,, \quad w(0) = w_0 \quad \text{a.e. in } \Omega\,; \tag{22}$$

the functions (u, z, w) comply with

- *unidirectionality and semistability*: for a.a. $x \in \Omega$, $z(\cdot, x)\colon [0,T] \to [0,1]$ is nonincreasing, cf. (2), and for a.a. $t \in (0,T)$ it holds

$$\forall\, \tilde{z} \in \mathcal{Z}\colon \quad \mathcal{E}(t,u(t),z(t)) \le \mathcal{E}(t,u(t),\tilde{z}) + \mathcal{R}_1(\tilde{z} - z(t))\,, \tag{23}$$

  where $\mathcal{Z}$ is defined in (7) and $\mathcal{E}(t,u,z)$ is given by (19);
- *weak formulation of the momentum equation*: for all $t \in [0,T]$

$$\begin{aligned}&\rho\int_\Omega \dot{u}(t)\cdot v(t)\,\mathrm{d}x - \rho\int_0^t\int_\Omega \dot{u}\cdot\dot{v}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega (\mathbb{D}(z,\Theta(w))e(\dot{u}) + \mathbb{C}(z)e(u) - \Theta(w)\,\mathbb{B}) : e(v)\,\mathrm{d}x\,\mathrm{d}s\\ &= \rho\int_\Omega \dot{u}_0\cdot v(0)\,\mathrm{d}x + \int_0^t \langle f, v\rangle_{H^1(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\end{aligned} \tag{24}$$

  for all test functions $v \in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3)) \cap W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^3))$;
- *mechanical energy equality*: for all $t \in [0,T]$

$$\begin{aligned}&\frac{\varrho}{2}\int_\Omega |\dot{u}(t)|^2\,\mathrm{d}x + \mathcal{E}(t,u(t),z(t)) + \int_\Omega (z_0 - z(t))\,\mathrm{d}x + \int_0^t\int_\Omega (\mathbb{D}(z,\Theta(w))e(\dot{u}) - \Theta(w)\,\mathbb{B}) : e(\dot{u})\,\mathrm{d}x\,\mathrm{d}s\\ &= \frac{\varrho}{2}\int_\Omega |\dot{u}_0|^2\,\mathrm{d}x + \mathcal{E}(0,u_0,z_0) + \int_0^t \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s - \int_0^t \langle f(t), \dot{g}\rangle_{H^1(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\\ &\quad + \rho\left[\int_\Omega \dot{u}(t)\cdot\dot{g}(t)\,\mathrm{d}x - \int_\Omega \dot{u}_0\cdot\dot{g}(0)\,\mathrm{d}x - \int_0^t\int_\Omega \dot{u}\cdot\ddot{g}\,\mathrm{d}x\,\mathrm{d}s\right]\\ &\quad + \int_0^t\int_\Omega (\mathbb{D}(z,\Theta(w))e(\dot{u}) + \mathbb{C}(z)e(u) - \Theta(w)\,\mathbb{B}) : e(\dot{g})\,\mathrm{d}x\,\mathrm{d}s\,;\end{aligned} \tag{25}$$

- *weak formulation of the enthalpy equation*: for all $t \in [0,T]$

$$\begin{aligned}&\langle w(t), \eta(t)\rangle_{W^{1,r'}(\Omega)} - \int_0^t\int_\Omega \Theta(w)\dot{\eta}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \mathbb{J}(z,w)\nabla w\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_\Omega w_0\,\eta(0)\,\mathrm{d}x + \int_0^t\int_\Omega \eta\,|\dot{z}|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega (\mathbb{D}(z,\Theta(w))e(\dot{u}) : e(\dot{u}) - \Theta(w)\,\mathbb{B}) : e(\dot{u})\eta\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_0^t\int_{\partial\Omega} h\eta\,\mathrm{d}\mathcal{H}^2(x)\,\mathrm{d}s + \int_0^t\int_\Omega H\eta\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{26}$$

  for all test functions $\eta \in W^{1,r'}(0,T;L^{r'}(\Omega)) \cap \mathrm{C}^0([0,T];W^{1,r'}(\Omega))$.

Observe that, since $r < \frac{5}{4}$, its conjugate exponent r' fulfills $r' > 5$. Hence the test functions η for (26) are continuous on $[0,T] \times \overline{\Omega}$, which makes them in duality with the measure $|\dot{z}|$. Notice that, for simplicity, we write $\int_0^t \int_\Omega \eta\, |\dot{z}|\; \mathrm{d}x\, \mathrm{d}s$ instead of $\iint_{(0,t)\times\Omega} \eta\, |\dot{z}|\, (\,\mathrm{d}s\, \mathrm{d}x)$.

We are now in the position to state our existence result.

Theorem 2.2. *Under assumptions* (4)–(6), (8) *on the datum* g, *and* (11) *on the data* f_{V}, f_{S}, H, *and* h, *for every quadruple* $(u_0, \dot{u}_0, z_0, \theta_0)$ *fulfilling* (9) *and* (10), *with* z_0 *satisfying* (23) *at* $t = 0$, *there exists an energetic solution* (u, z, w) *to the Cauchy problem for the enthalpy-reformulated system* (18) *such that*

$$w(x,t) \geq 0 \qquad \text{for a.a. } (t,x) \in (0,T) \times \Omega\,. \tag{27}$$

3. Proof of Theorem 2.2

The proof of the existence of an energetic solution to system (18) is based on time discretization, as for many results on rate-independent processes. We consider the solutions to carefully devised incremental problems and give the time-discrete version of the energetic formulation, consisting of the semistability, the weak momentum and enthalpy equations, and the (discrete) mechanical and total energy inequalities. However, due to the time-dependent Dirichlet loading g, we will not be able to deduce from the latter inequalities the basic a priori estimates on the approximate solutions, see also the comments at the beginning of Section 3.3. Therefore, we shall have to deduce a *further* energy inequality, see (48), allowing us to establish the first set of a priori estimates and thus to pass to the time-continuous limit by compactness. Finally, the properties of the energetic solutions at the time-continuous level will be obtained by passing to the limit in the corresponding discrete properties.

In what follows, we will focus on the steps needed to decouple the time-discrete scheme and to account for time-dependent Dirichlet conditions and develop the related calculations with some detail. The other parts of the proof of Theorem 2.2 will be only sketched here and we will systematically refer to [2] and [3].

3.1. Time-discrete scheme

Given a partition

$$0 = t_n^0 < \cdots < t_n^n = T \qquad \text{with} \qquad t_n^k - t_n^{k-1} = \tfrac{T}{n} =: \tau_n\,,$$

we construct a family of discrete solutions $(u_n^k, z_n^k, w_n^k)_{k=1}^n$ by solving the time-discretization scheme (33) below, where the data f, H, and h are approximated by *local means* as follows

$$f_n^k := \frac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} f(s)\, \mathrm{d}s\,, \qquad H_n^k := \frac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} H(s)\, \mathrm{d}s\,, \qquad h_n^k := \frac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} h(s)\, \mathrm{d}s\,, \tag{28}$$

(the above integrals understood in the Bochner sense).

Let us mention in advance that the main feature of system (33) is that the three equations are decoupled one from each other and can thus be solved recursively. This simplifies the analysis of (33) in comparison with [2], where the time-discrete versions of the momentum equation, of the flow rule, and of the enthalpy equations were coupled, and it was necessary to resort to a (non-constructive) fixed point argument to prove the existence of solutions.

In [2], the coupling between the discrete enthalpy and momentum equations served to ensure the nonnegativity of the discrete enthalpy w_n^k: this resulted from the (classical) test of the discrete enthalpy equation by $-(w_n^k)^-$, and to carry out the related calculations it was essential to have the thermal expansion term on the right-hand side *implicit* (i.e. with θ_n^k in place of the term θ_n^{k-1} that now features on the right-hand side of (33c)). Accordingly, the term θ_n^k appeared also in the discrete momentum equation to guarantee the cancelations leading to the discrete (mechanical) energy inequality.

In the present setting, we will be able to prove the nonnegativity of w_n^k by a different argument from the one in [2], based on the subquadratic growth (16) of Θ, cf. Lemma 3.3 below. Thanks to this, it will be possible to keep the discrete enthalpy and momentum equations, and ultimately the *whole* scheme, decoupled.

Nonetheless, because of the quadratic terms featuring on the right-hand side of the enthalpy equation (33c), which a priori is in $L^1(\Omega)$ only, it is necessary to introduce a regularization. One option, as done in [7], is to directly truncate the quadratic terms on the right-hand side of (33c), and then pass to the limit with the truncation parameter. Alternatively, as in [2], we here add a regularizing term of the form $-\tau_n \operatorname{div}(|e(u_n^k)|^2 e(u_n^k))$ to the discrete momentum equation (33b) below. This term ensures that the right-hand side of the discrete enthalpy equation (33c) is in $L^2(\Omega)$ and thus allows us to solve it by standard arguments for elliptic equations. Clearly, $-\tau_n \operatorname{div}(|e(u_n^k)|^2 e(u_n^k))$ will pass to zero with vanishing τ_n. In view of testing the momentum equation by $u-g$, this 4-Laplacian-term requires a regularization of the (extension on $[0,T]\times\Omega$ of the) Dirichlet datum g, and of $\tilde{u}_0 := u_0 - g(0)$. More precisely, using mollifiers as in [8, p. 56, Corollary 2], we approximate $\tilde{u}_0$ by a sequence

$$\tilde{u}_n^0 \in W_0^{1,4}(\Omega;\mathbb{R}^3) \ \text{ such that } \tilde{u}_n^0 \to \tilde{u}_0 \ \text{ in } H_0^1(\Omega;\mathbb{R}^3) \text{ as } n \to \infty \tag{29}$$

and, accordingly (cf. e.g. [9, p. 189] for appropriate mollifiers in time) the datum g by a sequence

$$\mathfrak{g}_n \in W^{1,4}(0,T;W^{1,4}(\Omega;\mathbb{R}^3)) \colon \begin{cases} \sup_{n\in\mathbb{N}} \tau_n^{1/4} \|\dot{\mathfrak{g}}_n\|_{L^4(0,T;W^{1,4}(\Omega;\mathbb{R}^3))} \le C\,, \\ \mathfrak{g}_n \to g \text{ in } H^1(0,T;H^1(\Omega;\mathbb{R}^3)) \cap W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))\,. \end{cases} \tag{30}$$

Using $\mathfrak{g}_n$, we then define the discrete data $(\mathfrak{g}_n^k)_{k=1}^n$ by setting

$$\mathfrak{g}_n^k := \mathfrak{g}_n(t_n^k) \qquad \text{for all } k = 1,\ldots,n. \tag{31}$$

Finally, for each $n \in \mathbb{N}$, the initial datum is given by $u_n^0 := \tilde{u}_n^0 + \mathfrak{g}_n(0)$.

Problem 3.1. *Starting from u_n^0, $z_n^0 := z_0$, $w_n^0 := w_0$, and $u_n^{-1} := u_0 - \tau_n \dot{u}_0$, find $(u_n^k, z_n^k, w_n^k)_{k=1}^n$ such that*

$$u_n^k \in W^{1,4}(\Omega;\mathbb{R}^3) \qquad \textit{with } u_n^k = \mathfrak{g}_n^k \textit{ on } \partial_{\mathrm{D}}\Omega\,, \tag{32a}$$

$$z_n^k \in W^{1,q}(\Omega)\,,\ \textit{cf.}\ (6\mathrm{d}), \tag{32b}$$

$$w_n^k \in H^1(\Omega)\,, \tag{32c}$$

and, denoting $\theta_n^{k-1} := \Theta(w_n^{k-1})$,

$$z_n^k \in \operatorname{argmin}\left\{ \int_\Omega (z_n^{k-1} - z)\,\mathrm{d}x + \mathcal{E}(t_n^k, u_n^{k-1}, z) \colon z \in W^{1,q}(\Omega)\,,\ 0 \le z \le z_n^{k-1} \le 1 \right\}, \tag{33a}$$

$$\begin{aligned} &\rho \int_\Omega \tfrac{u_n^k - 2u_n^{k-1} + u_n^{k-2}}{\tau_n^2} \cdot v \,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e\Big(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\Big) + \mathbb{C}(z_n^k) e(u_n^k) - \theta_n^{k-1}\,\mathbb{B} + \tau_n |e(u_n^k)|^2 e(u_n^k) \Big) : e(v)\,\mathrm{d}x \\ &= \Big\langle f_n^k, v \Big\rangle_{H^1(\Omega;\mathbb{R}^3)}\,, \end{aligned} \tag{33b}$$

where the above duality pairing again is to be understood in the sense of (12),

$$\begin{aligned}&\int_\Omega \tfrac{w_n^k-w_n^{k-1}}{\tau_n}\,\eta\,\mathrm{d}x+\int_\Omega \mathbb{J}(z_n^k,w_n^{k-1})\nabla w_n^k\cdot\nabla\eta\,\mathrm{d}x\\&=\int_\Omega\Big(\mathbb{D}(z_n^{k-1},\theta_n^{k-1})e\Big(\tfrac{u_n^k-u_n^{k-1}}{\tau_n}\Big)-\theta_n^{k-1}\mathbb{B}\Big):e\Big(\tfrac{u_n^k-u_n^{k-1}}{\tau_n}\Big)\,\eta\,\mathrm{d}x\\&\quad+\int_\Omega\tfrac{z_n^{k-1}-z_n^k}{\tau_n}\,\eta\,\mathrm{d}x+\int_{\partial\Omega}h_n^k\,\eta\,\mathrm{d}\mathcal{H}^2(x)+\int_\Omega H_n^k\,\eta\,\mathrm{d}x\,,\end{aligned}\tag{33c}$$

for every $v\in W_{\mathrm{D}}^{1,4}(\Omega;\mathbb{R}^3):=\{\tilde v\in W^{1,4}(\Omega;\mathbb{R}^3),\ \tilde v=0$ *on* $\partial_{\mathrm{D}}\Omega\}$ *and every* $\eta\in H^1(\Omega)$.

In (33a) the operator "argmin" generates the argument of the minimum, i.e. $\big(\int_\Omega(z_n^{k-1}-z_n^k)\,\mathrm{d}x+\mathcal{E}(t_n^k,u_n^{k-1},z_n^k)\big)=\min_{z\in W^{1,q}(\Omega)\,,\ 0\le z\le z_n^{k-1}\le 1}\big(\int_\Omega(z_n^{k-1}-z)\,\mathrm{d}x+\mathcal{E}(t_n^k,u_n^{k-1},z)\big)$.

We have the following existence result.

Lemma 3.2. *Assume* (4)–(6) *and* (8)–(11). *Then, for every* $n\ge 1$ *there exists a solution* $(u_n^k,z_n^k,w_n^k)_{k=1}^n$ *to Problem 3.1.*

Proof. The *existence of a minimizer for* (33a) follows from the Direct Method of the Calculus of Variations. Indeed, thanks to (6), the functional $z\mapsto\mathcal{R}_1(z-z_n^{k-1})+\mathcal{E}(t_n^k,u_n^{k-1},z)$ (with $\mathcal{R}_1$ from (20)) is coercive and (sequentially) weakly lower semicontinuous on $W^{1,q}(\Omega)$ (see the proof of [3, Prop. 3.2] for all the detailed calculations).

As for the *existence of solutions to* (33b), we observe that it is the Euler equation for the minimum problem

$$\begin{aligned}\min_{u\in\mathcal{A}_n^k}\Big\{&\tfrac{\varrho}{2}\int_\Omega\Big|\tfrac{u-2u_n^{k-1}+u_n^{k-2}}{\tau_n}\Big|^2\,\mathrm{d}x+\tfrac{\tau_n}{2}\int_\Omega\mathbb{D}(z_n^{k-1},\theta_n^{k-1})e\Big(\tfrac{u-u_n^{k-1}}{\tau_n}\Big):e\Big(\tfrac{u-u_n^{k-1}}{\tau_n}\Big)\,\mathrm{d}x\\&+\tfrac12\int_\Omega\mathbb{C}(z_n^k)e(u):e(u)\,\mathrm{d}x+\tfrac{\tau_n}{4}\int_\Omega|e(u_n^k)|^4\,\mathrm{d}x-\int_\Omega\theta_n^{k-1}\mathbb{B}:e(u)\,\mathrm{d}x-\Big\langle f_n^k,u\Big\rangle_{H^1(\Omega;\mathbb{R}^3)}\Big\}\end{aligned}\tag{34}$$

with $\mathcal{A}_n^k:=\{u\in W^{1,4}(\Omega;\mathbb{R}^3)\colon u=\mathfrak{g}_n^k$ on $\partial_{\mathrm{D}}\Omega\}$. The underlying functional is coercive and strictly convex on $W^{1,4}(\Omega;\mathbb{R}^3)$, hence the existence of a (unique) minimizer again ensues from the Direct Method.

Equation (33c), whose right-hand side is in $L^2(\Omega;\mathbb{R}^3)$, can be tackled by the same arguments as (33b). □

As previously mentioned, the subquadratic growth (16) of Θ is at the core of the proof of Lemma 3.3 below. Therein, the nonnegativity of the discrete enthalpy w_n^k is deduced by a *direct* argument that does not necessitate the implicit term θ_n^k on the right-hand side of (33c). Nonetheless, let us mention that a *strict positivity* result for w_n^k seems to be out of reach in the present context, while it is available with a fully implicit discrete enthalpy equation, cf. e.g. [10, Lemma 7.4] (the latter paper analyzing a temperature-dependent system for rate-independent adhesive contact).

Lemma 3.3. *Under assumptions* (4)–(6) *and* (8)–(11), *there exists* $\bar n\in\mathbb{N}$ *such that for all* $n\ge\bar n$ *we have* $w_n^k\ge 0$ *a.e. in* Ω *for every* $k=1,\ldots,n$.

Proof. We argue by induction. For all $n\in\mathbb{N}$ we have $w_n^0=w_0\ge 0$ a.e. in Ω by assumption. Consider now $n\in\mathbb{N}$ arbitrary but fixed. Suppose that $w_n^{k-1}\ge 0$. Taking $\eta=-(w_n^k)^-:=\max\{0,-w_n^k\}$ in (33c) we obtain

$$\int_\Omega\tfrac{((w_n^k)^-)^2}{\tau_n}\,\mathrm{d}x+\int_\Omega\Big(\mathbb{J}(z_n^k,w_n^{k-1})\nabla(w_n^k)^-\Big)\cdot\nabla(w_n^k)^-\,\mathrm{d}x$$

$$
\begin{aligned}
= & -\int_\Omega \left[\tfrac{w_n^{k-1}}{\tau_n} + \left(\mathbb{D}(z_n^{k-1},\theta_n^{k-1}) e\left(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\right) - \theta_n^{k-1}\,\mathbb{B}\right) : e\left(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\right)\right] (w_n^k)^- \,\mathrm{d}x \qquad (35)\\
& - \int_\Omega \tfrac{z_n^{k-1}-z_n^k}{\tau_n} (w_n^k)^- \,\mathrm{d}x - \int_{\partial\Omega} h_n^k (w_n^k)^- \,\mathrm{d}\mathcal{H}^2(x) - \int_\Omega H_n^k (w_n^k)^- \,\mathrm{d}x\,.
\end{aligned}
$$

We now remark that the left-hand side of the previous equality is nonnegative and the right-hand side is nonpositive. Indeed, $(w_n^k)^- \geq 0$, $z_n^{k-1} - z_n^k \geq 0$, $h_n^k \geq 0$, $H_n^k \geq 0$ a.e. in Ω, and

$$
\begin{aligned}
& \tfrac{w_n^{k-1}}{\tau_n} + \left(\mathbb{D}(z_n^{k-1},\theta_n^{k-1}) e\left(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\right) - \theta_n^{k-1}\,\mathbb{B}\right) : e\left(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\right)\\
& \geq \tfrac{w_n^{k-1}}{\tau_n} + C_1 \left| e\left(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\right)\right|^2 - \theta_n^{k-1}\,|\mathbb{B}| \left| e\left(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\right)\right|
\end{aligned}
$$

(recall that $\theta_n^{k-1} = \Theta(w_n^{k-1})$). The right-hand side of the last inequality is a nonnegative second-order polynomial in $\left| e\left(\frac{u_n^k - u_n^{k-1}}{\tau_n}\right)\right|$, since by growth condition (16)

$$
(\theta_n^{k-1}\,|\mathbb{B}|)^2 - 4\tfrac{w_n^{k-1}}{\tau_n} C_1 \leq \left((C_\Theta^2\,|\mathbb{B}|)^2 - 4\tfrac{C_1}{\tau_n}\right) w_n^{k-1} \leq 0
$$

for any $\tau_n = \frac{T}{n} \leq 4C_1/(C_\Theta^2\,|\mathbb{B}|)^2)$.

Hence, from (35) we deduce that $(w_n^k)^- = 0$ a.e. in Ω for every $n \in \mathbb{N}$ sufficiently large, whence the thesis. □

3.2. Approximate solutions and time-discrete version of the energetic formulation.
We now define the approximate solutions to (the energetic formulation of) the initial-boundary value problem for system (1) by suitably interpolating the discrete solutions $(u_n^k, z_n^k, w_n^k)_{k=1}^n$. We introduce the piecewise constant interpolants

$$
\begin{aligned}
\overline{u}_n(t) &:= u_n^k\,, \quad & \overline{w}_n(t) &:= w_n^k\,, \quad & \overline{z}_n(t) &:= z_n^k\,, \qquad (36a)\\
\underline{u}_n(t) &:= u_n^{k-1}\,, \quad & \underline{w}_n(t) &:= w_n^{k-1}\,, \quad & \underline{z}_n(t) &:= z_n^{k-1}\,, \qquad (36b)
\end{aligned}
$$

for $t \in (t_n^{k-1}, t_n^k]$, $k = 1, \ldots, n$, and the piecewise linear interpolants

$$
u_n(t) := \tfrac{t - t_n^{k-1}}{\tau_n} u_n^k + \tfrac{t_n^k - t}{\tau_n} u_n^{k-1}\,, \;\; z_n(t) := \tfrac{t - t_n^{k-1}}{\tau_n} z_n^k + \tfrac{t_n^k - t}{\tau_n} z_n^{k-1}\,, \;\; w_n(t) := \tfrac{t - t_n^{k-1}}{\tau_n} w_n^k + \tfrac{t_n^k - t}{\tau_n} w_n^{k-1}. \qquad (36c)
$$

We set $\overline{u}_n(0) = \underline{u}_n(0) = u_n(0) := u_0$, and analogously for $\overline{z}_n, \ldots, w_n$. We will use the notation $\underline{\theta}_n$ for $\Theta(\underline{w}_n)$.

We also introduce the piecewise constant and linear interpolants of the discrete data $(f_n^k, H_n^k, h_n^k)_{k=1}^n$ in (28) by setting for $t \in (t_n^{k-1}, t_n^k]$

$$
\overline{f}_n(t) := f_n^k\,, \qquad \overline{H}_n(t) := H_n^k\,, \qquad \overline{h}_n(t) := h_n^k\,,
$$

and $f_n(t) := \frac{t - t_n^{k-1}}{\tau_n} f_n^k + \frac{t_n^k - t}{\tau_n} f_n^{k-1}$ with time derivative $\dot{f}_n(t) := \frac{f_n^k - f_n^{k-1}}{\tau_n}$. It follows from (11) that, as $n \to \infty$,

$$
\begin{aligned}
& \overline{f}_n \to f \quad \text{in } L^p(0,T; H^1(\Omega;\mathbb{R}^3)^*) \;\; \text{for all } 1 \leq p < \infty\,,\\
& \overline{f}_n(t) \to f(t) \quad \text{in } H^1(\Omega;\mathbb{R}^3)^* \;\; \text{for all } t \in [0,T]\,, \qquad (37a)\\
& f_n \rightharpoonup f \quad \text{in } H^1(0,T; H^1(\Omega;\mathbb{R}^3)^*)\,, \qquad (37b)\\
& \overline{H}_n \to H \quad \text{in } L^2(0,T; L^2(\Omega))\,, \quad \overline{h}_n \to h \quad \text{in } L^2(0,T; L^2(\partial\Omega))\,. \qquad (37c)
\end{aligned}
$$

We shall denote by $\overline{\tau}_n$ the piecewise constant interpolant associated with the partition, i.e., $\overline{\tau}_n(t) := t_n^k$ for $t \in (t_n^{k-1}, t_n^k]$, with $\overline{\tau}_n(0) := 0$.

For the approximating discrete data of the Dirichlet loading $g \in H^1(0,T;H^1(\Omega;\mathbb{R}^3)) \cap W^{2,2}(0,T;L^2(\Omega;\mathbb{R}^3))$, cf. (30) and (31), observe that $\mathfrak{g}_n^k \in W^{1,4}(\Omega;\mathbb{R}^3)$ for all $k = 1,\ldots,n$: hence, $u_n^k - \mathfrak{g}_n^k - (u_n^{k-1} - \mathfrak{g}_n^{k-1})$ is an admissible test function for (33b). Thus, based on (31), we introduce the piecewise constant and linear interpolants

$$\overline{g}_n(t) := \mathfrak{g}_n^k, \quad \underline{g}_n(t) := \mathfrak{g}_n^{k-1}, \quad g_n(t) := \tfrac{t-t_n^{k-1}}{\tau_n}\mathfrak{g}_n^k + \tfrac{t_n^k - t}{\tau_n}\mathfrak{g}_n^{k-1} \tag{38}$$

for $t \in (t_n^{k-1}, t_n^k]$, $k = 1,\ldots,n$. Observe that, in order to make the notation more transparent we still use the letter g for the above interpolants, so that it will be clear that the functions $\overline{g}_n$, $\underline{g}_n$, g_n are approximations of g. In fact, arguing on a diagonal sequence it can be shown that

$$\begin{aligned} &\overline{g}_n(t) \to g(t) \quad \text{in } H^1(\Omega;\mathbb{R}^3) \ \text{ for all } t \in [0,T]\,, \\ &g_n \to g \quad \text{in } H^1(0,T;H^1(\Omega;\mathbb{R}^3))\,, \\ &\sup_{n\in\mathbb{N}} \tau_n^{1/4} \|\dot{g}_n\|_{L^4(0,T;W^{1,4}(\Omega;\mathbb{R}^3))} \le C\,. \end{aligned} \tag{39}$$

We shall also work with the piecewise linear interpolant of the values $\frac{\mathfrak{g}_n^k - \mathfrak{g}_n^{k-1}}{\tau_n}$, namely with $\gamma_n : [0,T] \to W^{1,4}(\Omega;\mathbb{R}^3)$ defined by

$$\gamma_n(t) := \tfrac{t-t_n^{k-1}}{\tau_n}\tfrac{\mathfrak{g}_n^k - \mathfrak{g}_n^{k-1}}{\tau_n} + \tfrac{t_n^k - t}{\tau_n}\tfrac{\mathfrak{g}_n^{k-1} - \mathfrak{g}_n^{k-2}}{\tau_n} \quad \text{for } t \in (t_n^{k-1}, t_n^k] \text{ and } k = 2,\ldots,n\,, \tag{40}$$

and $\gamma_n(t) := \frac{t}{\tau_n}\frac{\mathfrak{g}_n^1 - \mathfrak{g}_n^0}{\tau_n} + \frac{t_n^1 - t}{\tau_n}\frac{\mathfrak{g}_n^0 - \mathfrak{g}_n^{-1}}{\tau_n}$ with $\mathfrak{g}_n^{-1} := \mathfrak{g}_n^0 - \tau_n \dot{\mathfrak{g}}_n(0)$ for any $t \in [0, t_n^1]$. By construction we have $\dot{\gamma}_n(t) = \frac{\mathfrak{g}_n^k - 2\mathfrak{g}_n^{k-1} + \mathfrak{g}_n^{k-2}}{\tau_n^2}$ for all $t \in (t_n^{k-1}, t_n^k)$. Again, by (8), (30) and an argument along a diagonal sequence (also taking into account that $\|\gamma_n - \dot{g}_n\|_{L^\infty(0,T;L^2(\Omega))} \le \tau_n^{1/2}\|\dot{\gamma}_n\|_{L^\infty(0,T;L^2(\Omega))} \le C\tau_n^{1/2}$ to identify the limit of $(\gamma_n)_n$), one obtains

$$\gamma_n \to \dot{g} \ \text{ in } L^2(0,T;H^1(\Omega;\mathbb{R}^3)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))\,. \tag{41}$$

Proposition 3.4 below states that the approximate solutions constructed in the above lines indeed fulfill the discrete version of the energetic formulation. In the (discrete) mechanical energy inequality (44c) below, the mechanical energy $\mathcal{E}$ will be replaced by

$$\mathcal{E}_n(t,u,z) := \int_\Omega \left(\tfrac{1}{2}\mathbb{C}(z)e(u) : e(u) + \tfrac{\tau_n}{4}|e(u)|^4\right) \mathrm{d}x + \mathcal{G}(z,\nabla z) - \left\langle \overline{f}_n(t), u \right\rangle_{H^1(\Omega;\mathbb{R}^3)} \tag{42}$$

and we will understand the pointwise terms $\dot{u}_n$ and $\dot{g}_n$ as

$$\dot{u}_n(t) := \tfrac{u_n^k - u_n^{k-1}}{\tau_n}\,, \quad \dot{g}_n(t) := \tfrac{\mathfrak{g}_n^k - \mathfrak{g}_n^{k-1}}{\tau_n}\,, \quad \text{for } t \in (t_n^{k-1}, t_n^k], \quad \text{for } k = 1,\ldots,n\,. \tag{43}$$

Proposition 3.4. *Assume* (4)–(6) *and* (8)–(11). *Then the interpolants of the time-discrete solutions* $(\overline{u}_n, \underline{u}_n, u_n, \overline{z}_n, \underline{z}_n, z_n, \overline{w}_n, \underline{w}_n, w_n)$ *obtained via Problem 3.1 have the following properties:*

- *unidirectionality: for a.a.* $x \in \Omega$, *the functions* $\overline{z}_n(\cdot,x)\colon [0,T] \to [0,1]$ *are nonincreasing;*
- *discrete semistability: for all* $t \in [0,T]$

$$\forall\, \tilde{z} \in \mathcal{Z}\colon \quad \mathcal{E}_n(t, \underline{u}_n(t), \overline{z}_n(t)) \le \mathcal{E}_n(t, \underline{u}_n(t), \tilde{z}) + \mathcal{R}_1(\tilde{z} - \overline{z}_n(t))\,; \tag{44a}$$

- *discrete formulation of the momentum equation: for all* $t \in [0,T]$ *and for every* $(n+1)$*-tuple* $(v_n^k)_{k=0}^n \subset W_{\mathrm{D}}^{1,4}(\Omega;\mathbb{R}^3)$*, setting* $\overline{v}_n(s) := v_n^k$ *and* $v_n(s) := \frac{s-t_n^{k-1}}{\tau_n} v_n^k + \frac{t_n^k - s}{\tau_n} v_n^{k-1}$ *for* $s \in (t_n^{k-1}, t_n^k]$,

$$
\begin{aligned}
&\rho \int_\Omega \left(\dot{u}_n(t)\cdot\overline{v}_n(t) - \dot{u}_0\cdot v_n(0)\right)\mathrm{d}x - \rho\int_0^{\overline{\tau}_n(t)}\int_\Omega \dot{u}_n(s{-}\tau_n)\cdot\dot{v}_n(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^{\overline{\tau}_n(t)}\int_\Omega \big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n)+\mathbb{C}(\overline{z}_n)e(\overline{u}_n)-\underline{\theta}_n\,\mathbb{B}+\tau_n|e(\overline{u}_n)|^2 e(\overline{u}_n)\big):e(\overline{v}_n)\,\mathrm{d}x\,\mathrm{d}s\\
&=\int_0^{\overline{\tau}_n(t)}\left\langle \overline{f}_n,\overline{v}_n\right\rangle_{H^1(\Omega;\mathbb{R}^3)}\mathrm{d}s\,,
\end{aligned}
\tag{44b}
$$

where we have used $\underline{\theta}_n := \Theta(\underline{w}_n)$ *and extended* u_n *to* $(-\tau_n,0]$ *by setting* $u_n(t) := u_n^0 + t\dot{u}_0$*; again, the above duality pairing has the meaning of* (12)*;*

- *discrete mechanical energy inequality: for all* $t \in [0,T]$

$$
\begin{aligned}
&\tfrac{\rho}{2}\int_\Omega |\dot{u}_n(t)|^2\,\mathrm{d}x + \mathcal{E}_n(t,\overline{u}_n(t),\overline{z}_n(t)) + \int_\Omega (z_0{-}\overline{z}_n(t))\,\mathrm{d}x\\
&\quad+\int_0^{\overline{\tau}_n(t)}\int_\Omega \left(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n){-}\underline{\theta}_n\,\mathbb{B}\right):e(\dot{u}_n)\,\mathrm{d}x\,\mathrm{d}s\\
&\leq \tfrac{\rho}{2}\int_\Omega |\dot{u}_0|^2\,\mathrm{d}x + \mathcal{E}_n(0,u_n^0,z_0) - \int_0^{\overline{\tau}_n(t)}\left\langle \dot{f}_n,\underline{u}_n\right\rangle_{H^1(\Omega;\mathbb{R}^3)}\mathrm{d}s - \int_0^{\overline{\tau}_n(t)}\left\langle f_n,\dot{g}_n\right\rangle_{H^1(\Omega;\mathbb{R}^3)}\mathrm{d}s\\
&\quad+\rho\left[\int_\Omega \dot{u}_n(t)\cdot\dot{g}_n(t)\,\mathrm{d}x - \int_\Omega \dot{u}_0\cdot\dot{g}(0)\,\mathrm{d}x - \int_0^{\overline{\tau}_n(t)}\int_\Omega \dot{u}_n(s{-}\tau_n)\cdot\dot{\gamma}_n(s)\,\mathrm{d}x\,\mathrm{d}s\right]\\
&\quad+\int_0^{\overline{\tau}_n(t)}\int_\Omega \big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n)+\mathbb{C}(\overline{z}_n)e(\overline{u}_n)-\underline{\theta}_n\,\mathbb{B}+\tau_n|e(\overline{u}_n)|^2 e(\overline{u}_n)\big):e(\dot{g}_n)\,\mathrm{d}x\,\mathrm{d}s\,;
\end{aligned}
\tag{44c}
$$

- *discrete formulation of the enthalpy equation: for all* $t \in [0,T]$ *and for every* $(n+1)$*-tuple* $(\eta_n^k)_{k=0}^n \subset H^1(\Omega)$*, setting* $\overline{\eta}_n(s) := \eta_n^k$ *and* $\eta_n(s) := \frac{s-t_n^{k-1}}{\tau_n}\eta_n^k + \frac{t_n^k - s}{\tau_n}\eta_n^{k-1}$ *for* $s \in (t_n^{k-1}, t_n^k]$,

$$
\begin{aligned}
&\int_\Omega \overline{w}_n(t)\overline{\eta}_n(t)\,\mathrm{d}x - \int_\Omega w_0\,\eta_n(0)\,\mathrm{d}x - \int_0^{\overline{\tau}_n(t)}\int_\Omega \underline{w}_n(s)\dot{\eta}_n(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^{\overline{\tau}_n(t)}\int_\Omega \left(\mathbb{J}(\overline{z}_n,\underline{w}_n)\nabla\overline{w}_n\right)\cdot\nabla\overline{\eta}_n\,\mathrm{d}x\,\mathrm{d}s\\
&=\int_0^{\overline{\tau}_n(t)}\int_\Omega \overline{\eta}_n\,|\dot{z}_n|\,\mathrm{d}x\,\mathrm{d}s\int_0^{\overline{\tau}_n(t)}\int_\Omega \left(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n){-}\underline{\theta}_n\,\mathbb{B}\right):e(\dot{u}_n)\,\overline{\eta}_n\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^{\overline{\tau}_n(t)}\left[\int_{\partial\Omega}\overline{h}_n\,\eta_n\,\mathrm{d}\mathcal{H}^2(x) + \left\langle \overline{H}_n,\eta_n\right\rangle_{H^1(\Omega)}\right]\mathrm{d}s\,;
\end{aligned}
\tag{44d}
$$

- *discrete total energy inequality: for all* $t \in [0,T]$

$$
\begin{aligned}
&\tfrac{\rho}{2}\int_\Omega |\dot{u}_n(t)|^2\,\mathrm{d}x + \mathcal{E}_n(t,\overline{u}_n(t),\overline{z}_n(t)) + \int_\Omega \overline{w}_n(t)\,\mathrm{d}x\\
&\leq \tfrac{\rho}{2}\int_\Omega |\dot{u}_0|^2\,\mathrm{d}x + \mathcal{E}_n(0,u_n^0,z_0) + \int_\Omega w_0\,\mathrm{d}x\\
&\quad-\int_0^{\overline{\tau}_n(t)}\left\langle \dot{f}_n,\underline{u}_n\right\rangle_{H^1(\Omega;\mathbb{R}^3)}\mathrm{d}s - \int_0^{\overline{\tau}_n(t)}\left\langle f_n,\dot{g}_n\right\rangle_{H^1(\Omega;\mathbb{R}^3)}\mathrm{d}s
\end{aligned}
$$

$$+\rho\left[\int_\Omega \dot{u}_n(t)\cdot\dot{g}_n(t)\,\mathrm{d}x-\int_\Omega \dot{u}_0\cdot\dot{g}(0)\,\mathrm{d}x-\int_0^{\overline{\tau}_n(t)}\int_\Omega \dot{u}_n(s-\tau_n)\cdot\dot{\gamma}_n(s)\,\mathrm{d}x\,\mathrm{d}s\right]$$
$$+\int_0^{\overline{\tau}_n(t)}\int_\Omega\big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n)+\mathbb{C}(\overline{z}_n)e(\overline{u}_n)+\tau_n|e(\overline{u}_n)|^2e(\overline{u}_n)\big):e(\dot{g}_n)\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^{\overline{\tau}_n(t)}\left[\int_{\partial\Omega}\overline{h}_n\,\mathrm{d}\mathcal{H}^2(x)+\int_\Omega\overline{H}_n\,\mathrm{d}x\right]\mathrm{d}s\,. \tag{44e}$$

The proof of Proposition 3.4 closely follows the procedure developed in [2, Lemma 4.1], cf. also [3, Prop. 3.3] for the details with regard to our particular system. Let us here just hint at the main ideas, in particular dwelling on the treatment of the Dirichlet datum:

- The discrete semistability (44a) can be directly read from the minimality of z_n^k in (33a) tested by $\tilde{z}\le z_n^{k-1}$, also using that $\mathcal{R}_1(\tilde{z}-z_n^k)=+\infty$ if $\tilde{z}>z_n^k$. This property in particular enforces $z_n^k\le z_n^{k-1}$, whence unidirectionality.
- The discrete momentum and enthalpy equations (44b) and (44d) follow from (33b) and (33c), with test functions $(v_n^k)_{k=0}^n\subset W_{\mathrm{D}}^{1,4}(\Omega;\mathbb{R}^3)$ and $(\eta_n^k)_{k=0}^n\subset H^1(\Omega)$, respectively, applying the following discrete integration-by-parts formula, for every $(r_k)_{k=1}^n\subset X$ and $(s_k)_{k=1}^n\subset X^*$, with X a given Banach space (and $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X):

$$\sum_{k=1}^n\langle s_k,r_k-r_{k-1}\rangle_X=\langle s_n,r_n\rangle_X-\langle s_0,r_0\rangle_X-\sum_{k=1}^n\langle s_k-s_{k-1},r_{k-1}\rangle_X\,. \tag{45}$$

- The mechanical energy inequality (44c) results from summing (33a), tested by z_n^{k-1}, with the momentum balance (33b), tested by $v=u_n^k-\mathfrak{g}_n^k-u_n^{k-1}+\mathfrak{g}_n^{k-1}$. For the details of this calculation we refer to [3, Prop. 3.3]. Here we explain how the terms in (44c) involving the Dirichlet data (2nd and 3rd line of the RHS) emanate from (33b): Applying elementary convexity inequalities to (33b) tested by $v=u_n^k-\mathfrak{g}_n^k-u_n^{k-1}+\mathfrak{g}_n^{k-1}$ yields

$$\rho\int_\Omega\frac{u_n^k-2u_n^{k-1}+u_n^{k-2}}{\tau_n^2}\cdot(u_n^k-u_n^{k-1})\,\mathrm{d}x\ge\rho\int_\Omega\left(\frac{1}{2}\frac{|u_n^k-u_n^{k-1}|^2}{\tau_n^2}-\frac{1}{2}\frac{|u_n^{k-1}-u_n^{k-2}|^2}{\tau_n^2}\right)\mathrm{d}x\,, \tag{46a}$$
$$\int_\Omega\mathbb{C}(z_n^k)e(u_n^k):(e(u_n^k)-e(u_n^{k-1}))\,\mathrm{d}x\ge\int_\Omega\frac{1}{2}\Big(\mathbb{C}(z_n^k)e(u_n^k):e(u_n^k)-\mathbb{C}(z_n^k)e(u_n^{k-1}):e(u_n^{k-1})\Big)\,\mathrm{d}x, \tag{46b}$$
$$\int_\Omega\tau_n|e(u_n^k)|^2e(u_n^k):(e(u_n^k)-e(u_n^{k-1}))\,\mathrm{d}x\ge\int_\Omega\left(\frac{\tau_n}{4}|e(u_n^k)|^4-\frac{\tau_n}{4}|e(u_n^{k-1})|^4\right)\mathrm{d}x\,. \tag{46c}$$

By (38), the term $-\big(\mathbb{C}(z_n^k)e(u_n^k)+\mathbb{D}(z_n^k,\theta_n^k)e(\dot{u}_n)-\mathbb{B}\theta_n^{k-1}+\tau_n|e(u_n^{k-1})|^2e(u_n^{k-1})\big):(\mathfrak{g}_n^k-\mathfrak{g}_n^{k-1})$ results in the third line on the right-hand side of (44c). Further, let $t\in(0,T]$ be fixed, and let $1\le j\le n$ be such that $t\in(t_n^{j-1},t_n^j]$. We sum (46a)–(46c) over the index $k=1,\dots,j$. Applying the integration-by-parts formula (45) we conclude that

$$\begin{aligned}\sum_{k=1}^j\Big\langle f_n^k,u_n^k-u_n^{k-1}\Big\rangle_{H^1(\Omega;\mathbb{R}^3)}&=\int_0^{\overline{\tau}_n(t)}\langle\overline{f}_n,\dot{u}_n\rangle_{H^1(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\\&=\langle\overline{f}_n(t),\overline{u}_n(t)\rangle_{H^1(\Omega;\mathbb{R}^3)}-\langle f(0),u_0\rangle_{H^1(\Omega;\mathbb{R}^3)}-\int_0^{\overline{\tau}_n(t)}\Big\langle\dot{f}_n,\underline{u}_n\Big\rangle_{H^1(\Omega;\mathbb{R}^3)}\,\mathrm{d}s\,.\end{aligned} \tag{47}$$

Analogously, to deal with the term $\sum_{k=1}^j\rho\int_\Omega\frac{u_n^k-2u_n^{k-1}+u_n^{k-2}}{\tau_n^2}\cdot(\mathfrak{g}_n^k-\mathfrak{g}_n^{k-1})\,\mathrm{d}x$ we apply (45) with $s_k=\frac{g_n^k-g_n^{k-1}}{\tau_n}$, $r_k=\rho\frac{u_n^k-u_n^{k-1}}{\tau_n}$ and $r_{k-1}=\rho\frac{u_n^{k-1}-u_n^{k-2}}{\tau_n}$, which leads to the fifth, sixth, and seventh terms on the right-hand side of (44c).

- Finally, the discrete total energy inequality ensues from summing the discrete mechanical energy inequality (44c) with the discrete enthalpy equation (33c), tested for $\eta = \tau_n$ and added up over $k = 1, \dots, j$. Some terms cancel, leading to (44e).

3.3. A priori estimates

Usually, the first set of a priori estimates is deduced from the (discrete versions of the) mechanical and the total energy balance by a Gronwall argument exploiting the boundedness of the initial energy and of the power of the external loadings. However observe that, due to the time-dependent Dirichlet datum g, the right-hand sides of both (44c) and (44e) contain the term $\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : e(\dot{g}_n)$, which cannot be estimated if the viscous tensor $\mathbb{D}$ depends on (z, θ). Note, if $\mathbb{D}$ were constant, an integration by parts in time would allow us to control that term with $\int_0^{\overline{\tau}_n(t)} \|\underline{u}_n(s)\|^2_{H^1(\Omega;\mathbb{R}^3)} \,\mathrm{d}s$, under suitable conditions on $\ddot{g}$.

Instead, we have to develop an alternative estimate that contains $\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : e(\dot{g}_n)$ on both sides of the inequality, such that the term on the right can be absorbed by the corresponding one on the left-hand side. Hereby, the square-root growth of the enthalpy variable, cf. (16), generated by assumption (4a) on the heat capacity, will play a crucial role. More precisely, for the above described argument we sum up (44c) with (44d) tested by $\eta = \frac{\tau_n}{2}$ and obtain the second discrete total energy inequality, namely

$$
\begin{aligned}
&\tfrac{\rho}{2}\int_\Omega |\dot{u}_n(t)|^2 \,\mathrm{d}x + \mathcal{E}_n(t, \overline{u}_n(t), \overline{z}_n(t)) + \tfrac12 \int_\Omega (z_0 - \overline{z}_n(t))\,\mathrm{d}x \\
&\qquad + \tfrac12 \int_\Omega \overline{w}_n(t)\,\mathrm{d}x + \tfrac12 \int_0^{\overline{\tau}_n(t)} \int_\Omega (\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) - \underline{\theta}_n \mathbb{B}) : e(\dot{u}_n)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \tfrac{\rho}{2}\int_\Omega |\dot{u}_0|^2 \,\mathrm{d}x + \mathcal{E}_n(0, u_n^0, z_0) - \int_0^{\overline{\tau}_n(t)} \left\langle \dot{f}_n, \underline{u}_n \right\rangle_{H^1(\Omega;\mathbb{R}^3)} \mathrm{d}s - \int_0^{\overline{\tau}_n(t)} \left\langle \overline{f}_n, \dot{g}_n \right\rangle_{H^1(\Omega;\mathbb{R}^3)} \mathrm{d}s \\
&\qquad + \rho\left[\int_\Omega \dot{u}_n(t)\cdot \dot{g}_n(t)\,\mathrm{d}x - \int_\Omega \dot{u}_0 \cdot \dot{g}(0)\,\mathrm{d}x - \int_0^{\overline{\tau}_n(t)} \int_\Omega \dot{u}_n(s-\tau_n)\cdot \dot{\gamma}_n(s)\,\mathrm{d}x\,\mathrm{d}s \right] \\
&\qquad + \int_0^{\overline{\tau}_n(t)} \int_\Omega \big(\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) + \mathbb{C}(\overline{z}_n)e(\overline{u}_n) - \underline{\theta}_n \mathbb{B} + \tau_n |e(\overline{u}_n)|^2 e(\overline{u}_n) \big) : e(\dot{g}_n)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \tfrac12 \int_\Omega w_0\,\mathrm{d}x + \tfrac12 \int_0^{\overline{\tau}_n(t)} \left[\int_{\partial\Omega} \overline{h}_n \,\mathrm{d}\mathcal{H}^2(x) + \int_\Omega \overline{H}_n \,\mathrm{d}x \right] \mathrm{d}s
\end{aligned}
\tag{48}
$$

for all $t \in [0, T]$.

In the next lines we explain how to derive, starting from (48), estimates (49a)–(49g), cf. the forthcoming Proposition 3.5. For notational simplicity, we will use the symbols c, C to denote all the positive constants popping out in the following calculations, possibly varying from line to line. To control from below the left-hand side of (48) ($\doteq$ LHS(48)), first of all by Young's inequality and by the subquadratic growth (16) of Θ we get

$$
\left| \int_0^{\overline{\tau}_n(t)} \int_\Omega \underline{\theta}_n \mathbb{B} : e(\dot{u}_n)\,\mathrm{d}x\,\mathrm{d}s \right| \le \tfrac{1}{4\delta} \int_0^{\overline{\tau}_n(t)} \|\overline{w}_n(s)\|_{L^1(\Omega)} \,\mathrm{d}s + \delta \int_0^{\overline{\tau}_n(t)} \|e(\dot{u}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s\,,
$$

where δ is a positive constant that shall be chosen later. Hence, by the definition (42) of the mechanical energy $\mathcal{E}_n$, and by (5), (6), and (11a) we have

$$
\begin{aligned}
\mathrm{LHS}(48) \ge -C + \tfrac{\rho}{2} \|\dot{u}_n(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + C\left(\|\overline{u}_n(t)\|^2_{H^1(\Omega;\mathbb{R}^3)} + \tau_n \|\overline{u}_n(t)\|^4_{W^{1,4}(\Omega;\mathbb{R}^3)} + \|\overline{z}_n(t)\|^q_{W^{1,q}(\Omega)} \right) \\
+ \tfrac12 \|\overline{w}_n(t)\|_{L^1(\Omega)} + C \int_0^{\overline{\tau}_n(t)} \|e(\dot{u}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s
\end{aligned}
$$

$$- \tfrac{1}{4\delta} \int_0^{\overline{\tau}_n(t)} \|\overline{w}_n(s)\|_{L^1(\Omega)} \,\mathrm{d}s - \delta \int_0^{\overline{\tau}_n(t)} \|e(\dot{u}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s\,.$$

Here we have dropped the nonnegative term $\frac{1}{2}\int_\Omega (z_0 - \overline{z}_n(t))\,\mathrm{d}x$.

On the right-hand side of (48), the terms depending only on the initial and external data are uniformly bounded thanks to (9), (15), and (37). For the third summand, we use the Cauchy inequality as follows:

$$\left| \int_0^{\overline{\tau}_n(t)} \left\langle \dot{f}_n, \underline{u}_n \right\rangle_{H^1(\Omega;\mathbb{R}^3)} \,\mathrm{d}x \right| \le \tfrac{1}{2} \int_0^{\overline{\tau}_n(t)} \|\underline{u}_n(s)\|^2_{H^1(\Omega;\mathbb{R}^3)} \,\mathrm{d}s + \tfrac{1}{2} \int_0^{\overline{\tau}_n(t)} \left\|\dot{f}_n\right\|^2_{H^1(\Omega;\mathbb{R}^3)^*} \,\mathrm{d}s\,.$$

Moreover,

$$\begin{aligned}
&\int_\Omega \dot{u}_n(t) \cdot \dot{g}_n(t)\,\mathrm{d}x - \int_0^{\overline{\tau}_n(t)} \int_\Omega \dot{u}_n(s-\tau_n) \cdot \dot{\gamma}_n(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \tfrac{1}{4\delta} \|\dot{g}_n(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \delta \|\dot{u}_n(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + C \int_0^{\overline{\tau}_n(t)} \big(\|\dot{\gamma}_n(s)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\dot{u}_n(s-\tau_n)\|^2_{L^2(\Omega;\mathbb{R}^3)} \big)\,\mathrm{d}s\,.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_0^{\overline{\tau}_n(t)} \int_\Omega \big(\mathbb{D}(\underline{z}_n, \underline{\theta}_n) e(\dot{u}_n) + \mathbb{C}(\overline{z}_n) e(\overline{u}_n) - \underline{\theta}_n\, \mathbb{B} + \tau_n |e(\overline{u}_n)|^2 e(\overline{u}_n) \big) : e(\dot{g}_n)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \tfrac{1}{4\delta} \int_0^{\overline{\tau}_n(t)} \|e(\dot{g}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s + \delta \int_0^{\overline{\tau}_n(t)} \|e(\dot{u}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s \\
&\quad + C \int_0^{\overline{\tau}_n(t)} \|e(\overline{u}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s + C \int_0^{\overline{\tau}_n(t)} \|\overline{w}_n(s)\|_{L^1(\Omega)} \,\mathrm{d}s \\
&\quad + C\tau_n \left[\int_0^{\overline{\tau}_n(t)} \|e(\dot{g}_n(s))\|^4_{L^4(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s + \int_0^{\overline{\tau}_n(t)} \|e(\overline{u}_n(s))\|^4_{L^4(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s \right],
\end{aligned}$$

where we have used again (5) and the subquadratic growth (16) of Θ.

We then choose δ so small that the corresponding terms in the right-hand side are absorbed by larger terms on the left-hand side. Thus, taking into account the previous estimates and (39), from (48) we obtain

$$\begin{aligned}
&c\,\|\dot{u}_n(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} + c\|\overline{u}_n(t)\|^2_{H^1(\Omega;\mathbb{R}^3)} + \tau_n c\|\overline{u}_n(t)\|^4_{W^{1,4}(\Omega;\mathbb{R}^3)} + c\|\overline{z}_n(t)\|^q_{W^{1,q}(\Omega)} \\
&\quad + c\,\|\overline{w}_n(t)\|_{L^1(\Omega)} + c \int_0^{\overline{\tau}_n(t)} \|e(\dot{u}_n(s))\|^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})} \,\mathrm{d}s \\
&\le C + C \int_0^{\overline{\tau}_n(t)} \Big[\|\overline{w}_n\|_{L^1(\Omega)} + \|\underline{u}_n\|^2_{H^1(\Omega;\mathbb{R}^3)} + \|\dot{u}_n(s-\tau_n)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \tau_n \|\overline{u}_n\|^4_{W^{1,4}(\Omega;\mathbb{R}^3)} \Big]\,\mathrm{d}s\,,
\end{aligned}$$

where we have used the assumptions on initial and external data (8), (9), (10), and (11), and the constants c and C clearly depend on δ.

All in all, by suitable application of the Cauchy inequality and of the Gronwall Lemma, we obtain the a priori estimates (49a)–(49d), the $L^\infty(0,T;W^{1,q}(\Omega))$-bound on $\overline{z}_n$ in (49f), as well as estimate (49g) below, with a constant uniform with respect to n.

Relying on (49a)–(49d) and (49g), we are in the position to deduce from the discrete mechanical energy inequality (44c) a bound on the dissipation term $\int_\Omega (z_0 - \overline{z}_n(t))\,\mathrm{d}x$, which in particular ensures the $\mathrm{BV}(0,T;L^1(\Omega))$-bound on $\overline{z}_n$ in (49f). The $L^\infty((0,T)\times\Omega)$-bound on $\overline{z}_n$ is a direct consequence of the monotonicity and of the fact that $0 \le z_0 \le 1$.

Furthermore, estimate (49h) can be deduced using the Boccardo-Gallouët type estimates developed in [2, Prop. 4.2]. Let us just mention here, that it is based on testing the enthalpy equation (44d) by $\chi(\overline{w}_n) := 1 - 1/(1+\overline{w}_n)^\beta$ with $\beta > 0$, relying on assumptions (4) in order to handle the resulting gradient term. A chain of inequalities involving Gagliardo-Nirenberg estimates ultimately leads to the $L^r(0,T;W^{1,r}(\Omega))$-estimate in (49h) and the respective regularity of w in (21c). Finally, exploiting the estimates obtained so far, the BV-estimate in (49h) results from the boundedness of $\|\overline{w}_n\|_{L^1(0,T;W^{1,r'}(\Omega)*)}$, which is, in turn, deduced by a comparison argument in (44d) using test functions $\eta \in L^\infty(0,T;W^{1,r'}(\Omega)$ with $r' > 5$ by (21c) and the fact that the gradient term as well as the terms on the right-hand side of (44d) already have been proved to be bounded; we refer to [2] for the details.

Finally, estimate (49e) ensues from a comparison argument in the discrete momentum equation (44b), taking into account the previously obtained bounds (49a)–(49d), (49f)–(49h).

In total, the above arguments yield the following

Proposition 3.5 (A priori estimates). *Let the assumptions* (4)–(6) *and* (8)–(11) *hold true. Then a sequence of interpolants* $(\overline{u}_n, \underline{u}_n, u_n, \overline{z}_n, \underline{z}_n, z_n, \overline{w}_n, \underline{w}_n)_{n\in\mathbb{N}}$, *complying with the time-discrete version of the energetic formulation* (44), *satisfies*

$$\begin{aligned}
\|\overline{u}_n\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))} &\le C\,, && (49a)\\
\|\dot{u}_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} &\le C\,, && (49b)\\
\|\overline{u}_n\|_{L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^3))} &\le C/\sqrt[4]{\tau_n}\,, && (49c)\\
\|\dot{u}_n\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^3))} &\le C\,, && (49d)\\
\|\dot{u}_n\|_{\mathrm{BV}([0,T];W^{1,4}(\Omega;\mathbb{R}^3)^*)} &\le C\,, && (49e)\\
\|\overline{z}_n\|_{L^\infty(0,T;W^{1,q}(\Omega))\cap L^\infty((0,T)\times\Omega)\cap \mathrm{BV}([0,T];L^1(\Omega))} &\le C\,, && (49f)\\
\|\overline{w}_n\|_{L^\infty(0,T;L^1(\Omega))} &\le C\,, && (49g)\\
\|\overline{w}_n\|_{L^r(0,T;W^{1,r}(\Omega))\cap \mathrm{BV}(0,T;W^{1,r'}(\Omega)^*)} &\le C \quad \textit{for every } r < 5/4\,. && (49h)
\end{aligned}$$

3.4. Limit passage from time-discrete to time-continuous

A suitable version of Helly's selection principle, cf. e.g. [11, Thm. 6.1], combined with compactness arguments based on the a priori estimates (49a)–(49f), leads to the convergence statements (50a)–(50j) and (50o) in Proposition 3.6 below. We refer to [3, Prop. 4.1] for the details of the proof, here commenting only on the further convergence (50d), which follows from the bounds (49d) and (49e) via a BV-version of the Aubin-Lions compactness theorem, cf. e.g. [9, Cor. 7.9, pag. 196]. Moreover, convergences (50k)–(50m) for the enthalpy variables can be concluded from estimates (49g) and (49h) arguing along the lines of [2], cf. also [10]. Also based on (49g) and (49h), by the aforementioned Aubin-Lions type compactness result from [9], one additionally finds the strong convergence result (50n). From this, one may extract a further, pointwise a.e. convergent subsequence in order to see that the nonnegativity of the approximate solutions deduced in Lemma 3.3 carries over to the limit w for a.e. $t \in (0,T)$.

Proposition 3.6 (Convergence of the time-discrete solutions). *Let the assumptions* (4)–(6) *and* (8)–(11) *be satisfied. Then, there exists a triple* $(u,z,w)\colon [0,T]\times\Omega \to \mathbb{R}^3\times\mathbb{R}\times[0,+\infty)$ *of regularity* (21) *such that for a.a.* $x\in\Omega$ *the function* $t\mapsto z(t,x)\in[0,1]$ *is nonincreasing, nonnegativity* (27) *of* w *holds and there exists a subsequence of the time-discrete solutions* $(\overline{u}_n, \underline{u}_n, u_n, \overline{z}_n, \underline{z}_n, \overline{w}_n, \underline{w}_n)_n$ *from* (36) *such that*

$$\begin{aligned}
\overline{u}_n &\overset{*}{\rightharpoonup} u && \textit{in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))\,, && (50a)\\
u_n &\rightharpoonup u && \textit{in } H^1(0,T;H^1(\Omega;\mathbb{R}^3))\,, && (50b)
\end{aligned}$$

$$\dot{u}_n \overset{*}{\rightharpoonup} \dot{u} \quad \textit{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\,, \tag{50c}$$
$$\dot{u}_n \to u \quad \textit{in } L^2(0,T;W^{1-\varepsilon,2}(\Omega;\mathbb{R}^3))\,, \tag{50d}$$
$$\dot{u}_n(t) \rightharpoonup \dot{u}(t) \quad \textit{in } L^2(\Omega;\mathbb{R}^3) \textit{ for all } t\in[0,T]\,, \tag{50e}$$
$$\overline{z}_n\,,\, \underline{z}_n \overset{*}{\rightharpoonup} z \quad \textit{in } L^\infty(0,T;W^{1,q}(\Omega))\cap L^\infty((0,T)\times\Omega)\,, \tag{50f}$$
$$\overline{z}_n(t) \rightharpoonup z(t) \quad \textit{in } W^{1,q}(\Omega) \textit{ for all } t\in[0,T]\,, \tag{50g}$$
$$\overline{z}_n(t) \to z(t) \quad \textit{in } L^m(\Omega) \textit{ for all } m\in[1,\infty) \textit{ and for all } t\in[0,T]\,, \tag{50h}$$
$$\underline{z}_n(t) \rightharpoonup z(t) \quad \textit{in } W^{1,q}(\Omega) \textit{ for all } t\in[0,T]\backslash J\,, \tag{50i}$$
$$\underline{z}_n(t) \to z(t) \quad \textit{in } L^m(\Omega) \textit{ for all } m\in[1,\infty) \textit{ and for all } t\in[0,T]\backslash J\,, \tag{50j}$$
$$\overline{w}_n\,,\, \underline{w}_n \overset{*}{\rightharpoonup} w \quad \textit{in } L^\infty(0,T;L^1(\Omega))\,, \tag{50k}$$
$$\overline{w}_n\,,\, \underline{w}_n\,,\, w_n \rightharpoonup w \quad \textit{in } L^r(0,T;W^{1,r}(\Omega)) \quad \textit{for all } r<5/4\,, \tag{50l}$$
$$w_n(t) \overset{*}{\rightharpoonup} w(t) \quad \textit{in } W^{1,r'}(\Omega)^* \textit{ for all } t\in[0,T]\,, \tag{50m}$$
$$\overline{w}_n\,,\, \underline{w}_n\,,\, w_n \to w \quad \textit{in } L^r(0,T;W^{1-\varepsilon,r}(\Omega))\cap L^p(0,T;L^1(\Omega)) \tag{50n}$$

for all $\varepsilon\in(0,1]$ *and all* $p\in[1,\infty)$*. The set* $J\subset[0,T]$ *appearing in* (50i)–(50j) *denotes the jump set of* $z\in \mathrm{BV}([0,T];L^1(\Omega))$*. Finally,*

$$|\dot{z}_n| \to |\dot{z}| \quad \textit{in the sense of measures on } (0,T)\times\Omega\,. \tag{50o}$$

For such a limit triple (u,z,w), given by means of Proposition 3.6, it has to be verified that it solves the time-continuous energetic formulation stated in Def. 2.1. For this, one basically takes the limit $n\to\infty$ in the time-discrete energetic formulation (44). In what follows, we just outline the steps of the limit passage and comment on the main ideas and tools; for all the details the reader is referred to [3, Sect. 4], where the proof has been performed in an analogous setting.

- The *limit passage in the semistability inequality* can be carried out by verifying the *mutual recovery sequence condition*, cf. [11, 4], i.e. that for all $t\in[0,T]$, for any sequence $(v_n,\zeta_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \quad \zeta_n \rightharpoonup \zeta \text{ in } W^{1,q}(\Omega) \text{ and} \\ &\mathcal{E}_n(t,v_n,\zeta_n) \le \mathcal{E}_n(t,v_n,\hat{\zeta}_n) + \mathcal{R}_1(\hat{\zeta}-\zeta_n)\,, \end{aligned} \tag{51}$$

and for every $\tilde{\zeta}\in W^{1,q}(\Omega)$, there exists a mutual recovery sequence $(\tilde{\zeta}_n)_n$ such that

$$0 \le \limsup_{n\to\infty}\Big(\mathcal{E}_n(t,v_n,\tilde{\zeta}_n)-\mathcal{E}_n(t,v_n,\zeta_n)+\mathcal{R}_1(\tilde{\zeta}_n-\zeta_n)\Big) \le \mathcal{E}(t,v,\tilde{\zeta})-\mathcal{E}(t,v,\zeta)+\mathcal{R}_1(\tilde{\zeta}-\zeta)\,. \tag{52}$$

This condition is applied to the sequence $(v_n,\zeta_n)_{n\in\mathbb{N}} = (\overline{u}_n(t),\overline{z}_n(t))_n$, satisfying at every $t\in[0,T]$ the discrete semistability (44a) (whence (51)). The construction of the mutual recovery sequence $(\tilde{\zeta}_n)_{n\in\mathbb{N}}$ is developed in [3, Sect. 4.2], to which we refer for all details.
- For the *limit passage in the momentum balance* (44b) one considers test functions $v\in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3))\cap W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^3))$. Because of the regularization by the 4-Laplacian, such test functions have to be regularized using mollifers of the form [8, p. 56, Corollary 2] and [9, p. 189]. These mollifed functions are discretized in time according to (31), (38), and (43), thus resulting in admissible test functions $(\bar{v}_n,v_n)$ for (44b) satisfying for all $t\in[0,T]$

$$\begin{aligned} &\bar{v}_n(t), v_n(t) \to v(t) \text{ in } H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3)\,, \\ &\bar{v}_n \to v \text{ in } L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3)) \text{ and } v_n\to v \text{ in } L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^3))\cap W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^3))\,. \end{aligned}$$

Hence, in view of the weak convergences (50b)–(50e), to pass to the limit in (44b) weak-strong convergence arguments are employed. This is possible since

$$(\mathbb{D}(\underline{z}_n, \Theta(\underline{\theta}_n)) + \mathbb{C}(\underline{z}_n))e(\bar{v}_n) \to (\mathbb{D}(z, \Theta(w)) + \mathbb{C}(z))e(v) \quad \text{in } L^2(0, T, L^2(\Omega; \mathbb{R}^{3\times 3}))\,,$$

which can be deduced via dominated convergence; we refer to [3, Sect. 4.1] for the details.

- The *mechanical energy equality* (25) for the limit system is proved in two steps. The part "$\leq$" directly follows from (44c) by lower semicontinuity for the terms on the left-hand side, and by weak-strong convergence arguments for the terms on the right-hand side, making use of the convergence properties of the interpolants of the given data (29), (30), (37), (39), and (43), which, in particular, also ensure energy convergence at initial time.
- The *opposite mechanical energy inequality* "$\geq$" is deduced following the lines of [2, p. 283f]: A Riemann sum argument is applied to the already proven semistability inequality (23) of the limit (u, z, w) at times t_{i-1}, $i \in \{1, \ldots, n\}$ tested with $z(t_i)$. The result is added to the momentum balance (24) of the limit tested by $(\dot{u} - \dot{g})$. This can be done rigorously in view of the enhanced $H^1(0, T; H^1(\Omega; \mathbb{R}^3))$-regularity of $\dot{u}$, which can be gained by a comparison argument in the momentum balance, cf. [3, Rmk. 2.6] for all details. Combined with the previously deduced inequality "$\leq$", this yields the *mechanical energy balance* (25) of the limit system, see [3, Prop. 4.6] for the details. Using a lim sup-argument, i.e. starting from the discrete energy inequality (44c) and passing over to the limit in a chain of inequalities, the fact that *equality* holds for the limit system additionally allows us to conclude the *strong convergence of the viscous dissipation terms* in $L^1([0, T] \times \Omega; \mathbb{R}^{3\times 3})$ by comparison.
- Finally, we carry out the *limit passage in the enthaly equation*. Starting with a function $\eta \in W^{1,r'}(0, T; L^{r'}(\Omega)) \cap \mathrm{C}^0([0, T]; W^{1,r'}(\Omega))$, also in this step, time-discrete test functions $(\bar{\eta}_n, \eta_n)$ for (44d) are constructed as in (31) and (38) and thus exhibit strong convergence in $W^{1,r'}(0, T; L^{r'}(\Omega)) \cap L^{r'}(0, T; W^{1,r'}(\Omega))$. Via dominated convergence, also using convergence (50n) and the boundedness (17) of $\mathbb{J}$, it can be shown that

$$\mathbb{J}(\bar{z}_n, \underline{\theta}_n)^* \nabla \bar{\eta}_n \to \mathbb{J}(z, w)^* \nabla \eta \quad \text{strongly in } L^{r'}(0, T; L^{r'}(\Omega, \mathbb{R}^3))\,,$$

which is a major ingredient to obtain the *enthalpy equation* (26) of the the limit system. The strong convergence of the viscous dissipation terms in $L^1([0, T] \times \Omega; \mathbb{R}^{3\times 3})$ ultimately enables us to pass to the limit in the right-hand side of (44d). We refer to [10, p. 30] for more details.

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